Portfolio Revision: A Turnover-Constrained Approach

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Introduction

This article presents a new approach to portfolio revision that attempts to build a bridge between the portfolio selection models of academicians and the work of practicing portfolio managers.

Portfolio management is a broad activity which may involve more finance professionals than is frequently realized. It is surely the major activity of a fund manager; in addition, it is often a significant activity of corporate chief financial officers and their staffs. For example, it is usually desirable to maintain among divisions of the firm and among capital investment projects a portfolio balance not unlike that maintained by a fund manager among security investments. In addition, the chief financial officer is frequently responsible for the firm's investment portfolios, as well as for its pension and profit-sharing funds. On the side, he may be a college endowment fund trustee or a mutual fund director. All these activities involve updating, or revising, existing portfolios to adjust for changing conditions and new information.

The pioneering work of Markowitz and the succeeding developments of most other researchers apply, for the most part, to the more basic task of portfolio *selection*, which is the determination of a group of securities for initial investment.

The related but broader process of portfolio *revision* has developed slowly. This is felt to be one of two major bottlenecks currently impeding progress and implementation of scientific portfolio management. (The other is the slow development of a comfortable interface between security analysts and portfolio analysts, whereby the former produce statistical estimates of the future performance of individual securities and of the market for use in models of the latter.)

Nudging the revision process ahead is the goal of this paper. It presents a portfolio revision methodology that appears to be theoretically proper, potentially understandable to both academicians and practitioners, computationally feasible, and operationally inexpensive.

The methodology builds on the portfolio selection theory of Markowitz, summaries of which are widely available, such as in Francis and Archer [6] and Cohen and Pogue [2].

Portfolio Revision to Date

Previous revision models [1, 4, 10, 16, 18] have tended to be difficult to program and expensive to compute, as well as to require burdensome estimation of transactions costs. Even approximate determination of those costs — such as brokerage commissions, illiquidity costs, and taxes — is complex and may involve computationally difficult non-linear and/or discontinuous expressions [1, p. 53, and 10, pp. 1009–1011]. They may vary with transaction size, but in inconsistent directions. For example, as transaction size increases, commission cost per unit tends to decrease, but the unit cost of unfavorable price movements when trading in imperfect markets tends to increase. In addition, the hazy but real expenses of research and decision time ought somehow to be included.

The model presented in this paper abandons the cost approach as impractical as to both computation and implementation. A different concept, *turnover rate*, is proposed as a more appropriate and useful decision criterion for portfolio revision. A portfolio model with a constraint on turnover rate can be readily programmed to achieve useful results. Further, and helpful for implementation, this concept is already familiar to practitioners.

The Concept of Turnover

Rate of portfolio turnover is generally measured as the proportion of a portfolio's securities that changes in the revision process. If, for example, Portfolio 1 consisting of:

IBM	$ 4,000
GM	4,000
AT&T	2,000
	$10.000

is revised by sale of GM and purchase of Exxon, to result in Portfolio 2:

IBM	$ 4,000
Exxon	4,000
AT&T	2,000
	$10.000

it is clear that 40% of the securities have been changed, or "turned over."

Turnover rate is measured per period, usually per year. During any time period, individual securities typically fluctuate unevenly in value. Suppose that Portfolio 1 represents holdings on January 1; that transactions during the year are:

Sold: AT&T for $2,200
Bought: Exxon for $2,200;

and that December 31 holdings have these market values (Portfolio 3):

IBM	$ 3,900
GM	5,200
Exxon	2,900
	$12,000

In this instance, portfolio turnover rate, PTR, is:

$$PTR = \frac{\text{value sold}}{\text{average portfolio value}} = \frac{\text{value bought}}{\text{average portfolio value}} = \frac{2{,}200}{11{,}000} = 20\%.$$

The most general case must also provide for intraperiod cash flows into and/or out of the portfolio. Let Portfolio 1 again be the January 1 holdings, and assume also the $2,200 trade from AT&T to Exxon. But at another time during the year, a commitment requires selling enough IBM to raise $1,500. The resulting December 31 holdings are (Portfolio 4):

IBM	$2,600
GM	4,500
Exxon	2,300
	$9,400

What is the effect of this sale of IBM on turnover rate? There is none, because turnover rate is intended as a measure of portfolio changes determined by revised expectations of securities' performance. Changes so determined typically involve a relatively simultaneous sale of one set of securities and purchase of another set of similar value. Turnover rate is not affected by an excess of purchases over sales or vice versa due to an exogenous inflow or outflow of cash. Accordingly, portfolio turnover rate is finally defined as purchases or sales during the period, whichever amount is smaller, divided by average total value. For the example, total purchases are $2,200, total sales $3,700. Using the smaller figure, purchases:

$$PTR = \frac{2{,}200}{9{,}700} = 22.7\%$$

Mathematically:

$$PTR = \frac{(P + S - NI)/2}{(A_1 + A_2)/2} \qquad (1)$$

where: PTR = portfolio turnover rate for a given period; P = total purchases of portfolio securities during the period; S = total sales of portfolio securities during the period; NI = absolute value of net inflow or outflow of money during the period; A_1 = net assets at the beginning of the period; and A_2 = net assets at the end of the period.

This definition is in virtual agreement with the method prescribed in Securities and Exchange Commission [12] for use by mutual funds. (The only difference, a minor one, is the SEC's use of monthly average total value in the denominator.) And it is identical with the turnover formula used in the well-known mutual fund study of the Wharton School [19, p. 210].

Rate of portfolio turnover is a measurable quantity and an accepted concept. Its definition is not controversial. It is widely used both formally and informally by investment practitioners. And so it is a natural, logical candidate for service in portfolio revision models. We will move to putting it to work for that purpose after a brief discussion on notation.

Notation for Portfolio Revision

Let X be a vector consisting of the proportionate values of individual securities in a portfolio. More specifically, let X_t^- represent a portfolio just prior to revision at time t, and let X_t^+ represent the portfolio just after revision.

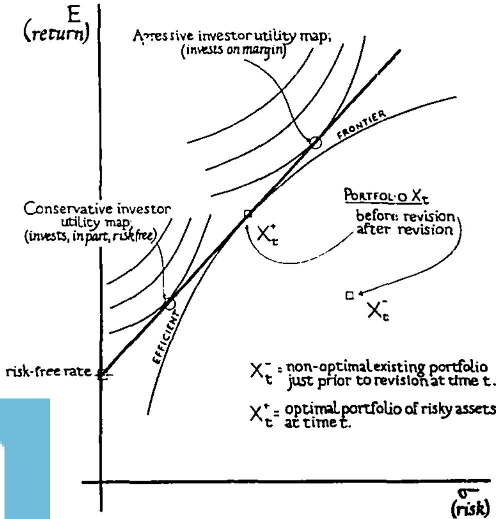
At some previous time t-1, portfolio X_{t-1}^+ was chosen for investment. (The choice then was either an initial selection, usually the result of cash investment, or a revision of an even earlier portfolio from time t-2.) Since previous time t-1, and present time t, there will have been two significant developments. First, security price movements will have changed the proportions indicated by vector X_{t-1}^+. Those securities that have gone up relatively more (or down relatively less) are a greater fraction of the total, and vice versa. As a result, portfolio X_{t-1}^+ has become portfolio X_t^-. Second, expectations for individual securities will have changed, rendering X_{t-1}^+ (which has become X_t^-) inefficient, as shown in Exhibit 1. A currently efficient portfolio, X_t^+, is desired.

If turnover from existing securities to new securities is unrestricted, and if transactions costs are negligible, revised portfolio X_t^+ can be found by traditional techniques of initial portfolio selection. But typically, transactions costs are not negligible, and other turnover restrictions exist. Accordingly, the general portfolio revision problem is to identify a new portfolio that maximizes investor utility after taking turnover costs and constraints into account.

Incorporating the Turnover Constraint

The portfolio selection model (in any of its various forms) can be broadened into a portfolio revision model by constraining turnover of the existing portfolio to a designated maximum rate. (It is of signif-

Exhibit 1. The Revision Challenge



icant note that the portfolio selection model is quite sensitive to changes in input data and that this results in typical turnover rates at or near 100%. For most practitioners, such rates are unacceptably high.)

Mathematically speaking, the portfolio selection model finds minimum variance at all levels of expected return (or, equivalently, maximum expected return at all levels of variance). The revision model adds a requirement or constraint on maximum turnover. (Constraints are frequently encountered and readily handled in mathematical programming models. For example, the portfolio selection model includes a constraint requiring that the "x" variables sum to one. This paper merely adds an additional constraint — albeit a somewhat complex one — to the list.)

From a practical standpoint, this added constraint may somewhat simulate the thinking of the professional portfolio manager. Ideally, he might prefer to substitute a completely new portfolio for the existing one, but instead he makes only selective changes because of taxes, commissions, and other costs of high turnover. A further restraining influence on managers of some portfolios such as mutual funds is a requirement to disclose portfolio turnover rates. High rates are conventionally viewed with disfavor.

After transformation into the notation of mathematical programming models, the definition of turnover in Equation (1) becomes:

$$PTR_t = \frac{\Sigma \mid x^+_{it} - x^-_{it} \mid}{2} \qquad (2)$$

where: PTR_t = portfolio turnover rate at time t; x^+_{it} = proportion of security i in the portfolio just after revision at time t; and x^-_{it} = proportion of security i in the existing portfolio just prior to revision at time t.

It is assumed in this paper that there are no external cash inflows or outflows. An extension to consider these is not difficult, but to make assumptions here on such flows would unnecessarily complicate this discussion.

It is also assumed that the revision process is conducted once per period. Accordingly, Equation (2) does not provide for averaging two asset values in the denominator as does Equation (1). This is consistent with other academic studies including those referenced earlier. Of course, portfolio managers in the real world practice their art continuously rather than at discrete time points. And they must, in addition, consider matters such as numbers of shares, round lots, minor transfers to and from the cash account to the extent that values of purchases and sales mis-match, and others. While all these considerations are important real-world details, they also can be easily and most efficiently handled separately.

Turnover as defined in Equation (2) can be incorporated as a constraint into any of the several mean-variance portfolio selection models — the full variance-covariance model, the various multi-index models, or the single index model. This paper demonstrates use of the latter of these.

To eliminate the awkward absolute value signs, Equation (2) is revised to an equivalent:

$$PTR_t = \frac{\Sigma\, c_i(x^+_{it} - x^-_{it})}{2} \qquad (3)$$

where: c_i shifts between +1 and −1 so that $c_i = +1$ when $x^+_{it} > x^-_{it}$, and $c_i = -1$ when $x^+_{it} < x^-_{it}$.

Provision must also be made for the turnover constraint to shift from binding to non-binding and vice versa. This is necessary because, as will be recalled, the constraint designates a *maximum* turnover rate. Where turnover in an unconstrained case is less than the maximum, the constraint must not be binding.

Incorporating the turnover constraint into a computer program of the portfolio selection model proved to be a fully logical, yet lengthy mathematical and programming task. The appendix presents a brief description of this process.

Results in Simple Situations

Exhibits 2, 3, and 4 present efficient frontiers for a small population of five artificially designated securities. In Exhibit 2, the existing or pre-revision portfolio is invested 100% in security 1. Exhibits 3 and 4 present efficient frontiers where the existing portfolio is 100% invested in stocks 3 and 5, respectively. In each exhibit, separate curves of efficient frontiers represent maximum turnover rates of 25%, 50%, 75%, and 100%. The last of these cases, 100% turnover, is, of course, the completely unrestrained case — identical with results from the initial portfolio selection model. It should be noted that certain portions of the curves represent the turnover constraint in binding mode, others in non-binding mode.

Results in a Major Simulation

The new revision model operates efficiently on large populations of securities as well as small, and at all rates of turnover. This broad ability is demonstrated

Exhibit 2. Revision Possibilities at Various Maximum Turnover Rates. $X_t^- \equiv$ Stock 1

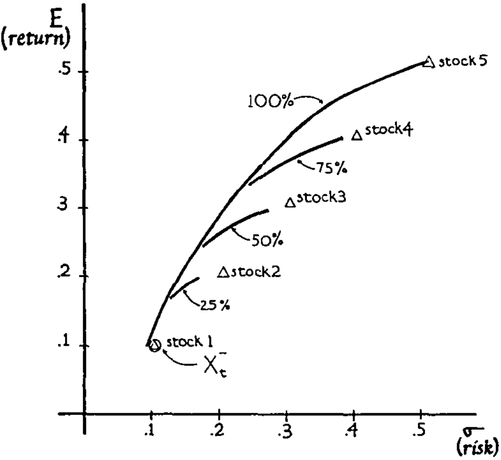


Exhibit 3. Revision Possibilities at Various Maximum Turnover Rates. $X_t^- \equiv$ Stock 3

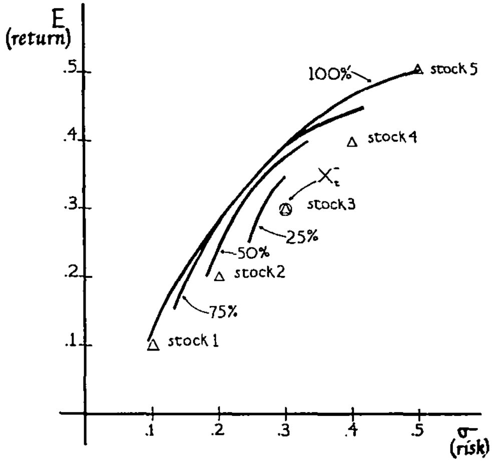
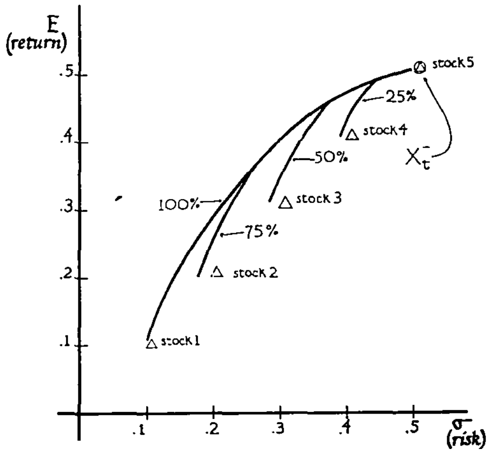


Exhibit 4. Revision Possibilities at Various Maximum Turnover Rates. $X_t^- \equiv$ Stock 5



by reporting the results of the following major simulation:

1. Basic simulation plan: time period one decade, 1966–1975; initial portfolio just prior to December 31, 1965, invested equally in each of the population's securities, and then revised on December 31, 1965, and annually thereafter through December 31, 1974; in each year, the newly-revised portfolio lies "unmanaged" from January 1 to December 31; cash dividends received on any security reinvested in that security.

2. Population: common stocks of the 50 largest NYSE firms measured by aggregate common stock market value at beginning of the simulated decade; for the market index, the New York Stock Exchange Composite Index.

3. Forecast of investment performance: *ex ante* expectations for each security and for the market forecast at the end of each year by extrapolating the previous five years' actual annual performance. (The simulations of [11] show that *ex ante* expectations based on the most recent five or so annual historical returns generally produce the best *ex post* portfolio performance.)

4. Permissible turnover rate: separate simulations conducted for maximum annual turnover rates of 100%, 75%, 50%, 25%, and 0%.

5. Choice of efficient portfolio: the tangent portfolio associated with a 5% risk-free lending/borrowing rate chosen each December 31 for investment during the next year; this intended to represent roughly the utility map of an "intermediate" investor — one whose preferences are midway between conservative and aggressive.

6. Performance measurement: for each year, total return (representing both capital gains/losses and dividends) noted; then two measures of *ex post* performance for the decade determined: geometric mean return and reward to variability ratio (see Sharpe [13]).

7. Income taxes: a tax-free situation assumed, corresponding to the real-world setting for foundations,

university endowment funds, pension funds and, basically, for mutual funds.

Exhibit 5 shows the simulation in diagrammatic form.

This research design leads to a multi-period horizon produced from a series of single-period optimizations. Fama [3] shows that, under these conditions, the multi-period result is also optimal.

Exhibit 6 presents detailed results for the case of 25% maximum turnover, while Exhibit 7 gives overall results for the five designated turnover limits.* Where 100% turnover is permitted, the outcome is identical to that produced by the initial portfolio selection model, as in Schreiner [11]. In the 0% case, with no turnover at all, the initial portfolio (invested equally in all available stocks) runs ten years without external revision. Of course, those stocks performing relatively well over the years become larger proportions of the total portfolio and vice versa.

It should be evident that composition of the initial portfolio is increasingly crucial as rate of permissible turnover is decreased. For this paper, it was arbitrarily decided that the initial set of securities be passively chosen, with equal investment in all those available. Prominent among numerous other possibilities would be an efficient set determined by the initial selection model (or, equivalently, by the revision model with 100% turnover permitted).

Exhibit 7 indicates that, for this simulation, moderate turnover rates are associated with the most favorable performance. In particular, at 25% maximum turnover, the highest measures are achieved for both geometric mean return and reward to variability ratio. This is worth noting, even though it is not a strong conclusion, given the arbitrary choice of initial portfolio and the naive, mechanical method of generating *ex ante* expectations — via extrapolations of history. In the same vein, and subject to the same limitations, one cannot fail to notice that performance at 0% turnover, a buy-and-hold strategy, exceeds that of all three cases of 50% or higher turnover. These results favor conservative strategies of buy/hold or moderate turnover, and disfavor so-called churning. Interestingly, this accords with the conventional wisdom of most practitioners.

Among the variations that others may wish to explore is an analysis combining turnover constraints and estimated transaction costs. This might be approached by plotting utility-maximizing portfolios under a series of maximum turnover rates, as in Exhibit 8. One could then analyze transactions costs while moving along a gradient line, as shown, to chosen portfolios representing steadily increasing turnover rates. The optimal rate would be where expected return less costs is highest.

*The average central processor time for a ten-year simulation with one turnover rate on a population of 50 securities is about 8 seconds on Control Data Corporation's Cyber 75 system. This is about twice the corresponding time for the initial portfolio selection model (and producing the same results as in the 100% turnover case). The increased time is felt reasonable in view of the more realistic and useful results. The Cyber 75's commercial rate is on the order of $16 per central processor minute, so the cost for 8 seconds of roughly $2 is well within a nominal range.

Exhibit 5. The Multi-Year Simulation Plan

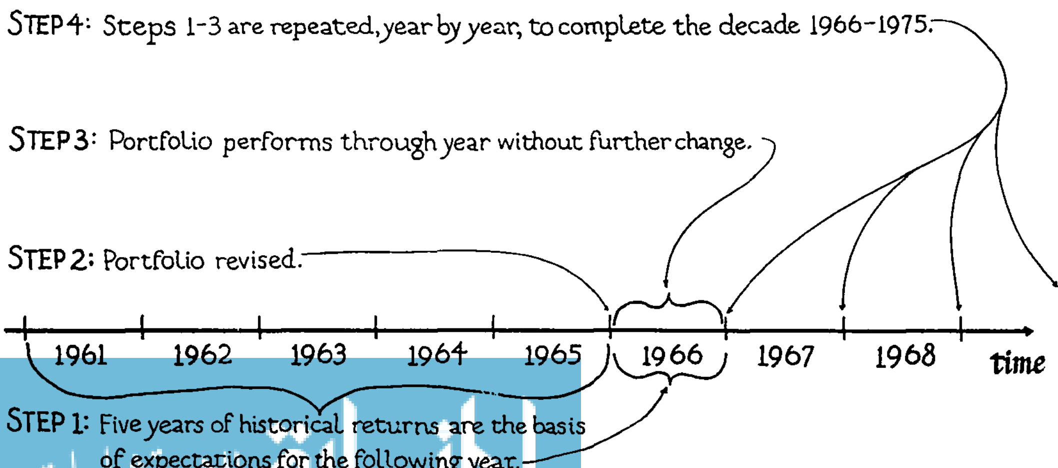


Exhibit 6. Ten-Year Simulation, Detailed Results. (50 stocks, maximum turnover 25% per year.)

Year	Actual Turnover from Previous Year	No. of Stocks In Portfolio	Year's Investment Return	Cumulative Value of $10,000 Invested at 12/31/65, at End of Year
1966	25%	38	- 1.76%	$ 9,824
1967	25%	23	5.21%	10,336
1968	25%	21	49.62%	15,465
1969	25%	20	- 6.86%	14,405
1970	25%	17	-10.35%	12,914
1971	25%	14	27.94%	16,522
1972	25%	13	29.48%	21,392
1973	25%	15	-17.30%	17,691
1974	25%	14	-27.70%	12,791
1975	25%	10	19.62%	15,300

Arithmetic mean return	6.79%	Of 10 periods, turnover constraint binding in	10
Standard Deviation	22.99%	Average annual turnover	25.0%
Geometric Mean Return	4.34%	Average number of stocks invested	18
Reward to Variability Ratio	.0779		

Exhibit 7. Ten-Year Simulation, Summary Results. (50 stocks, investment in years 1966–1975.)

	Performance Measures								
Maximum Turnover Per Year	Arithmetic Mean Return	Standard Deviation	Ending Value of Initial $10,000	Geometric Mean Return	Reward to Variability Ratio	Of 10 Periods, Constraint Binding In	Average Annual Turnover	Average No. Of Stocks Invested	Computer Processing Time (sec.)
100%	4.70%	25.03%	$11,900	1.75%	-.0121	0	60.6%	11	8.3
75%	4.55%	25.18%	$11,680	1.57%	-.0179	1	60.3%	10	7.7
50%	5.92%	25.19%	$13,276	2.87%	.0364	8	49.4%	13	7.9
25%*	6.79%	22.99%	$15,300**	4.34%**	.0779**	10	25 %	18	7.3
0%	6.08%	19.45%	$15,091	4.20%	.0557	10	0 %	50	2.9

*Exhibit 6 gives detailed results for this case.
**Best value in each column of performance measures.

Summary and Conclusions

This paper presents a new approach to revising portfolios which limits turnover of securities to a designated maximum rate. It is demonstrated that this is readily achievable — mathematically, computationally and economically — for populations of securities from small to large. The method produces theoretically proper results without resort to judgmental searching or testing. Further, it sidesteps the difficult task of estimating total trading costs by substituting a constraint describable in terms familiar to both academicians and practitioners. One of the model's key advantages is that it can be easily understood by, and discussed between, both of these groups.

References

1. A. H. Y. Chen, F. C. Jen, and S. Zionts, "The Optimal Portfolio Revision Policy," *Journal of Business* (January 1971), pp. 51–61.
2. K. J. Cohen and J. A. Pogue, "An Empirical Evaluation of Alternative Portfolio Selection Models," *Journal of Business* (April 1967), pp. 166–193.
3. E. F. Fama, "Multiperiod Consumption-Investment Decisions," *American Economic Review* (March 1970), pp. 163–174.
4. L. Fisher, "Using Modern Portfolio Theory To Maintain an Efficiently Diversified Portfolio," *Financial*

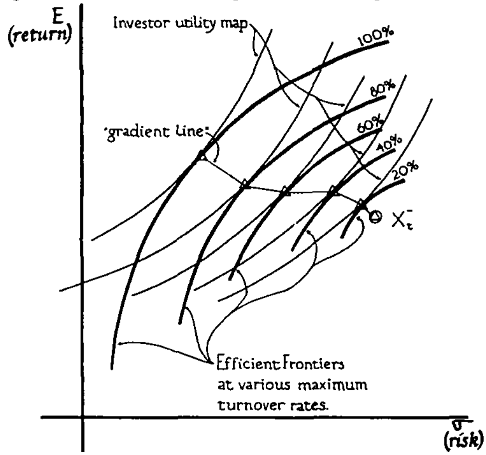
Analysts Journal (May–June 1975), pp. 73–85.
5. J. C. Francis, *Investments, Analysis and Management*, 2nd ed., New York, McGraw-Hill Book Co., 1976, Ch. 18.
6. J. C. Francis and S. H. Archer, *Portfolio Analysis*, Englewood Cliffs, N.J., Prentice-Hall, 1971, Ch. 6.
7. H. A. Latané, D. L. Tuttle, and C. P. Jones, *Security Analysis and Portfolio Management*, 2nd ed., New York, Ronald Press, 1975, Ch. 20.
8. H. M. Markowitz, "Portfolio Selection," *Journal of Finance* (March 1952), pp. 77–91.
9. H. M. Markowitz, *Portfolio Selection: Efficient Diversification of Investments*, New York, John Wiley & Sons, 1959.
10. J. A. Pogue, "An Extension of the Markowitz Portfolio Selection Model to Include Variable Transactions' Costs, Short Sales, Leverage Policies, and Taxes," *Journal of Finance* (December 1970), pp. 1005–1028.
11. J. Schreiner, "Historically Successful Portfolio Selection Rules," unpublished Ph.D. dissertation, University of California, Los Angeles, 1970.
12. Securities and Exchange Commission, *General Instructions for Form N-1R, Annual Report of Registered Management Investment Company Under the Investment Company Act of 1940 and the Securities Exchange Act of 1934*, Washington, D.C., 1971, Item 1.25 — Portfolio Turnover Rates.
13. W. F. Sharpe, "Mutual Fund Performance," *Journal of Business* (January 1966), pp. 119–138.
14. W. F. Sharpe, *Portfolio Theory and Capital Markets*, New York, McGraw-Hill Book Co., 1970.
15. W. F. Sharpe, "A Simplified Model for Portfolio Analysis," *Management Science* (January 1963), pp. 277–293.
16. K. V. Smith, "Alternative Procedures of Revising Investment Portfolios," *Journal of Financial and Quantitative Analysis* (December 1968), pp. 371–403.
17. K. V. Smith, *Portfolio Management*, New York, Holt, Rinehart and Winston, 1971.
18. K. V. Smith, "A Transition Model for Portfolio Revision," *Journal of Finance* (September 1967), pp. 425–439.
19. Wharton School of Finance and Commerce, *A Study of Mutual Funds*, prepared for the Securities and Exchange Commission, Washington, D.C., U.S. Government Printing Office, 1962.

Appendix. Mathematical and Programming Details

The procedure for tracing the efficient frontier of the turnover-constrained revision model is sketched here mathematically and descriptively.

Several comparisons with the less complex portfolio selection model are made. Readers interested in greater technical detail with regard to that model are referred to Markowitz [9], Schreiner [11], and Sharpe [15]. Those desiring computer programs in FORTRAN language for either model are invited to write the author.

Exhibit 8. Proposed Extension. Combine expected return and turnover costs at various points along the "gradient line" to find optimal revised portfolio



The portfolio selection model is, mathematically, a problem of quadratic programming. In vector notation, the simplified diagonal selection model of Sharpe [15] is:

Minimize $X'QX$

Subject to:

$$\sum^{n} x_i = 1$$

$$\sum^{n} x_i b_i = b_{n+1}$$

$$\sum^{n+1} a_i x_i = E^*$$

$$x_i \geqq 0, i = 1, 2, 3, \ldots, n$$

This becomes the revision model discussed in this paper by adding a turnover constraint formed from Equation (3):

$$\sum^{n} \frac{c_i\,(x_i^+ - x_i^-)}{2} \leq \text{TOL} \qquad \text{(A-1)}$$

where TOL = portfolio turnover limit.

After moving the constants to the right side, this becomes:

$$\sum^{n} c_i x_i^+ \leq 2\,\text{TOL} + \sum^{n} c_i x_i^-. \qquad \text{(A-2)}$$

The frontier of efficient portfolios is traced as this problem is solved for values of portfolio expected return, E*, ranging from the highest possible E* down to the E* of the minimum variance portfolio. This is done first, by locating the highest E* corner portfolio, and second, by finding succeeding corner portfolios, one by one.

After applying the LaGrangian technique to the constraints, the following algebraic matrix expression is formed:

$$\left[\begin{array}{cccccc|ccc} q_1 & & & & & & 1 & b_1 & c_1 \\ & q_2 & & 0\text{'s} & & & 1 & b_2 & c_2 \\ & & q_3 & & & & 1 & b_3 & c_3 \\ & & & \ddots & & & \vdots & \vdots & \vdots \\ & 0\text{'s} & & & q_n & & 1 & b_n & c_n \\ & & & & & q_{n+1} & 0 & -1 & 0 \\ \hline 1 & 1 & 1 & \cdots & 1 & 0 & & & \\ b_1 & b_2 & b_3 & \cdots & b_n & -1 & & 0\text{'s} & \\ c_1 & c_2 & c_3 & \cdots & c_n & 0 & & & \end{array}\right] \left[\begin{array}{c} x_1^+ \\ x_2^+ \\ x_3^+ \\ \vdots \\ x_n^+ \\ x_{n+1}^+ \\ \hline \lambda_a \\ \lambda_b \\ \lambda_c \end{array}\right] = \left[\begin{array}{c} \\ \\ 0\text{'s} \\ \\ \\ \hline 1 \\ 0 \\ RHS_{TO} \end{array}\right] + \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \\ a_{n+1} \\ \hline 0 \\ 0 \\ 0 \end{array}\right] \lambda_E$$

where: $RHS_{TO} = 2\,TOL + \sum^{n} c_i x_i^-$ and λ_E = E-V slope.

In each interval between corner portfolios, this procedure is applied to those variables designated as "in," meaning available to vary in value. Variables designated as "out" in a given interval are fixed in value at upper or lower limits.

Finding the First Corner Portfolio

For initial portfolio selection, the first corner portfolio is 100% invested in that security with the highest expected return, E. If the upper investment limit of that security is less than 100%, then the remainder is invested in the security of second-highest E, and so on.

In the case of turnover-constrained portfolio revision, the first corner portfolio is found as follows: find the highest E security currently invested below its upper limit and the lowest E security currently invested above its lower limit (*i.e.*, above zero). Increase investment in the highest E security and simultaneously reduce investment in the lowest E security until the resulting turnover increases to the designated maximum. If the respective limit for either stock is reached before the resulting turnover reaches the designated maximum, shift to the *next* highest/lowest E stock, respectively, and so on.

Finding Succeeding Corner Portfolios

For initial portfolio selection under the Markowitz critical line method, each "out" security of the preceding corner portfolio is tested to find the E-V slope at which it would come "in;" and each "in" security is tested to find the E-V slope at which it would go "out." The succeeding corner portfolio is at the highest slope just found which is less than or equal to the slope of the preceding corner portfolio.

In the case of turnover-constrained portfolio revision, the testing is more extensive. The succeeding corner portfolio is at the highest E-V slope which is less than or equal to the slope of the preceding corner portfolio, as found by the following tests:

1. For each "out" security, find the slope at which it would come "in;" for each "in" security, find the slope at which it would go "out."
2. For each "in" security, find the slope at which its proposed investment proportion ("x_i^+" value) would equal its investment proportion in the existing portfolio ("x_i" value). The purpose of a corner portfolio chosen by this test is to reverse the "c" value from (−1) to (+1) (or vice versa) as is required by the turnover constraint's basic definition. See Equations (3) and (A-1).
3. If the turnover constraint is binding, find the slope at which it would no longer bind. The purpose of a corner portfolio chosen by this test is to de-activate the turnover constraint. Conversely, if the turnover constraint is non-binding, find the slope at which turnover increases to the designated maximum. The purpose of a corner portfolio chosen by this test is to activate the turnover constraint, and thereby prevent turnover from exceeding the designated limit.

The process concludes when the minimum variance portfolio is found, as identified by a zero slope.